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M.A./M.Sc. FIRST SEMESTER EXAMINATION, MARCH 2022
FIRST YEAR [BATCH 2021-23]
Mathematics

Date : 23/03/2022 **Real Analysis 1**
Time : 11.00 am – 1.00 pm **Paper** : MTM CC3 Full Marks:50

Answer any **five** questions, each question carries 10 marks.

1. Let A and B be separated subsets of some $\mathbb{R}^k$. Suppose $a \in A, b \in B$, and define
$$p(t) = (1-t)a + tb$$
for $t \in \mathbb{R}^1$. Put $A_0 = p^{-1}(A)$, $B = p^{-1}(B)$.
(a) Prove that A_0 and B_0 are seperated subsets of $\mathbb{R}^1$.
(b) Prove that there exists $t_0 \in (0,1)$ such that $p(t_0) \notin A \cup B$.
(c) Prove that every convex subset of $\mathbb{R}^k$ is connected.

2. Let E be a dense subset of a metric space X and let f be uniformly continuous function defined on E and taking value in a complete metric space Y. Prove that f has a unique continuous extension from E to X.

3. (a) Suppose K and F are disjoint sets in a metric space such that K is compact and F is closed. Then
$$d(K,F) = \inf\big\{d(x,y) : x \in K, y \in F\big\} > 0.$$
(b) Prove that the Cauchy product of two absolutely convergent series converges absolutely.

4. Let (X,d) be a metric space and $E \subseteq X$. Then consider the function
$$\rho_E : X \longrightarrow \mathbb{R}_{\geq 0}$$
where $\rho_E(x) = d(x,E) = \inf\big\{d(x,y) : y \in E\big\}$. Prove that (i) $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.
(ii) ρ_E is uniformly continuous.
Let A and B be two disjoint closed sets. Then consider the function $f : X \longrightarrow [0,1]$
$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$
for all $x \in X$. Then Prove that (i) f is continuous on X, (ii) $f|_A = 0$ and $f|_B = 1$.

5. Let (X,d) be a metric space. We call two Cauchy sequences $\{p_n\}$ and $\{q_n\}$ in X equivalent, denoted by $\{p_n\} \sim \{q_n\}$ if $\lim\limits_{n\to\infty} d(p_n,q_n) = 0$.
(a) Prove that $\sim$ is an equivalence relation.
(b) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*, Q \in X^*$, $\{p_n\} \in P, \{q_n\} \in Q$, define
$$\Delta(P,Q) = \lim_{n\to\infty} d(p_n,q_n).$$
Show that the number $\Delta(P,Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^*.
(c) Prove that (X^*,Δ) is a complete metric space.

6. (a) Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

   (b) Prove that every compact metric space K has a countable base, and that K is therefore separable.

7. (a) Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \qquad \big(a < x < b\big).$$

   (b) Suppose

   (i) f is continuous for $x \geq 0$.

   (ii) $f'(x)$ exists for $x > 0$

   (iii) $f(0) = 0$

   (iv) f' is monotonically increasing.

   Put

$$g(x) = \frac{f(x)}{x} \text{ for } x > 0,$$

   and prove that g is monotonically increasing.

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